RAPID COMMUNICATIONS

Influence of long-range interactions on the critical behavior of systems with a negative Fisher exponent

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The influence of long-range interactions decaying in d dimensions as $1/R^{d+\sigma}$ on the critical behavior of systems with *negative* Fisher's correlation-function exponent for short-range interactions, $\eta_{SR}<0$, is reexamined. Such systems, typically described by ϕ^3-field theories, are, e.g., the Potts model in the percolation limit, the Edwards-Anderson spin-glass model, and the Yang-Lee edge singularity. In contrast to preceding studies, it is shown by means of Wilson's momentum-shell renormalization-group recursion relations that the long-range interactions dominate as long as $\sigma<2-\eta_{SR}$. Exponents change *continuously* to their short-range values at the boundary of this region. [S1063-651X(98)50609-5]

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Fifteen years ago there was some debate about the influence of long-range interactions [decaying as $J(R)\sim 1/R^{d+\sigma}$ in d dimensions] on the critical exponents of systems that show a *negative* Fisher-exponent η_{SR} if the long-range forces are absent [1,2]. Finally it was claimed that the long-range interactions leading to a Fisher-exponent $\eta_{LR}=2-\sigma$ dominate for all $\sigma<2$, and, by reason of an instability, the exponents change discontinuously to their short-range values at $\sigma=2$. The assumption that long-range interactions decaying with $\sigma>2$ are generally equivalent to purely short-range interactions seems to be the accepted lore. Indeed in the Gaussian part of an effective Landau-Ginzburg-Wilson Hamiltonian in momentum space $\mathcal{H}_0=\frac{1}{2}\int_q (r+j_2 q^2+j_\sigma q^\sigma +\cdots)s_q s_{-q}$ (s_q is the Fourier transform of an order parameter fluctuation and $\int_q\cdots=\int d^d q\cdots$) where as usual j_2 stems from the short-range part of the interaction (which also contributes to r) and j_σ from the long-range one, j_σ is naively irrelevant in comparison to j_2 at long-wave lengths if $\sigma>2$. Of course, this is an incorrect argument below the upper critical dimension where the nonquadratic higher-order terms of the Hamiltonian play a dominant role. The relevance of the long-range term $\propto q^\sigma$ (which does not renormalize because of its nonanalyticity in q) has to be determined by comparison with the scaling behavior of the full inverse correlation function $\Gamma_2(q)_{SR}\propto q^{2-\eta_{SR}}$ that corresponds to the nontrivial stable fixed point solution of a properly chosen renormalization-group transformation.

In the present Rapid Communication I show that also in the case $\eta_{SR}<0$ the long-range interaction dominates as long as $2-\sigma=\eta_{LR}>\eta_{SR}$. At $\eta_{LR}=\eta_{SR}$ the exponents change *continuously* to their short-range values that hold everywhere for $\sigma>2-\eta_{SR}$. This behavior is well known for models with $\eta_{SR}>0$ [3–7]. In the case $\eta_{SR}<0$ the incorrect result of [1,2] with respect to the crossover has not been corrected so far. It arises from a renormalization group that is not appropriate for that case because it mixes in a redundant operator in the terminology of Wegner [8]. I show that in general the limit $\sigma\rightarrow 2$ produces a contribution $\sim q^2\ln q$ to the Gaussian part of the Hamiltonian that is a relevant perturbation here. Thus, the critical exponents are different for the cases with or without such a perturbation that is therefore responsible for the apparent discontinuity of exponents.

Systems with a negative Fisher exponent η_{SR} are typically described by critical ϕ^3-field theoretic models [9] with an upper critical dimension $d_c=6$, such as the Potts model in the percolation limit [10–12], the Yang-Lee singularity

model [13,14], and the Edwards-Anderson spin-glass model in the replica formalism [15,16]. Besides these equilibrium models, nonequilibrium models of the ϕ^3 type are given by epidemic processes that lead to percolation clusters. Here long-range interactions creep in if the disease spreads by Levy flights [17,18]. While treating these nonequilibrium models I became aware of the earlier work [1,2] on long-range interactions in ϕ^3 models.

In this Rapid Communication I concentrate on the discussion of the Yang-Lee model as the simplest among all others leading to similar recursion equations. To stick close to the above mentioned work, I use the Wilson renormalization-group transformation based on the elimination of short-wave fluctuations and a hard momentum cutoff normalized to $q_c = 1$. I perform the renormalization group to one-loop order and use the ε expansion where $\varepsilon = 6 - d$. This is sufficient to produce nontrivial crossover behavior, since $\eta_{SR} = O(\varepsilon)$.

I write the Yang-Lee Hamiltonian of the scalar field s in the following form:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} (\nabla s)^2 + \frac{v}{4\rho} [(\nabla^{1-\rho} s)^2 - (\nabla s)^2] + \frac{r}{2} s^2 + \frac{ig}{6} s^3 + ihs \right\}. \tag{1}$$

Here $(\nabla^{1-\rho} s)^2$ is defined in momentum space as $q^{2(1-\rho)} s_q s_{-q}$, and the momentum scale is chosen such that $|q| \leq q_c = 1$. I have written the long-range exponent as $\sigma = 2(1-\rho)$. Then $\rho = O(\varepsilon)$ in the crossover region. The Gaussian part of $\mathcal{H}$ with the derivatives of the field is positive definite as long as $v \geq 0$ irrespective of the sign of ρ, and reads in momentum space $\frac{1}{2} q^2 (1 - v \ln q) s_q s_{-q}$ in the limit $\rho \to 0$. Thus, also in this limit the model does not coincide with its short-ranged counterpart unless $v = O(\rho)$. The unperturbed correlation function (the propagator of a diagrammatic perturbation expansion) follows from Eq. (1) as

$$G_0(q) = \left(q^2 + \frac{v}{2\rho} [q^{-2\rho} - 1] q^2 + r \right)^{-1}. \tag{2}$$

Thus, the scale of the fields is defined such that for $r = 0$ the propagator is 1 at the cutoff momentum $q_c = 1$. The propagator is positive definite for all q and ρ as it should be for stability.

The calculation of the momentum integrals that arise by eliminating to one-loop order fluctuations s_q which depend on momenta in the interval $b^{-1} < q \leq 1$ with $b \approx 1$ is standard and does not present any technical difficulties. The coefficients of the terms of different order in the Hamiltonian $\mathcal{H}$ in Eq. (1) change by the elimination procedure to

$$q^2 \left\{ 1 + \frac{v}{2\rho} (q^{-2\rho} - 1) \right\} + r$$

$$\to q^2 \left\{ 1 + \frac{v}{2\rho} (q^{-2\rho} - 1) - \frac{2wK(v,r)}{d(1+r)^4} \ln b \right\}$$

$$+ \left\{ r + \frac{w}{(1+r)^2} \ln b \right\} + O(q^4, w^2) \tag{3}$$

and

$$g \to g \left\{ 1 - \frac{2w}{(1+r)^3} \ln b + O(q^2, w^2) \right\}, \tag{4}$$

$$h \to h + \frac{w/g}{1+r} \ln b. \tag{5}$$

Here, I have defined

$$w = (4\pi)^{-d/2} g^2 / \Gamma(d/2), \tag{6}$$

and $K(v,r)$ is given by

$$K(v,r) = \frac{d-2}{4} + \frac{2+2\rho-d}{8} v - \frac{v^2}{8} - \frac{1-\rho}{4} vr + \frac{2-v}{8} dr. \tag{7}$$

After elimination of the short-wavelength fluctuations an appropriate rescaling has to be introduced as the last step of the renormalization transformation. The goal is to choose a rescaling ζ of the fields $s^<(\mathbf{r}) = \zeta s'(b^{-1}\mathbf{r})$, where $s^<(\mathbf{r}) = \int_{q \leq b^{-1}} e^{i\mathbf{q} \cdot \mathbf{r}} s_q$ in such a way that a renormalization group is constructed which leads to fixed points and does not mix in *redundant dangerous operators* [8]. The operator generated by a rescaling of the fields without an elimination is such a redundant operator. Thus, the rescaling factor ζ must be chosen carefully. The old (and working) definition for ζ follows from the requirement to hold the propagator finite at the cutoff momentum to exclude infrared singularities. Then such singularities cannot arise in the full elimination procedure. Note that in the long-range interaction problem normally one holds constant the coefficient of the nonanalytic term $\sim q^{2(1-\rho)}$. This may be the simplest possibility but leads to the difficulty of a vanishing inverse propagator in the present case [1,2]. Therefore, I define

$$\zeta^2 = b^{2-d-\psi(w,v)}. \tag{8}$$

Here the function $\psi(w,v)$ follows from the requirement $G_0'(q=1, r'=0) = G_0(q=1, r=0) = 1$ for the rescaled propagator. Now by rescaling Eqs. (3) and (4), renormalized parameters are found from

$$q^2 \left\{ 1 + \frac{v'}{2\rho} (q^{-2\rho} - 1) \right\}$$

$$= q^2 b^{-\psi} \left\{ 1 + \frac{v}{2\rho} (b^{2\rho} q^{-2\rho} - 1) - \frac{2wK(v,r)}{d(1+r)^4} \ln b \right\}, \tag{9}$$

and

$$r' = b^{2-\psi} \left\{ r + \frac{w}{(1+r)^2} \ln b \right\}, \tag{10}$$

$$w' = w b^{6-d-3\psi} \left\{ 1 - \frac{4w}{(1+r)^3} \ln b \right\}, \tag{11}$$

$$h' = b^{(d+2-\psi)/2} \left\{ h + \frac{w/g}{1+r} \ln b \right\}. \qquad (12)$$

Taking Eq. (9) for $q=1$ yields the equation for ψ,

$$1 = b^{-\psi} \left\{ 1 + \frac{v}{2\rho}(b^{2\rho}-1) - \frac{2wK(v,r)}{d(1+r)^4} \ln b \right\}. \qquad (13)$$

Expanding $b = \exp l$ to first order in l around $l=0$ one finds

$$\psi = v - \frac{2wK(v,r)}{d(1+r)^4}$$

$$= v - \frac{w}{3} \left(1 - \frac{2-\rho}{4} v - \frac{v^2}{8} \right) + O(\varepsilon^2). \qquad (14)$$

In the last equation I have used $r = O(w)$, $w = O(\varepsilon)$, and I have retained only terms linear in ε. Note that the dependence on v is exact to linear order in w. For an infinitesimal transformation with $l=dl$ one gets the renormalization group equations from the remaining parts of the Eqs. (9)–(12) as

$$\frac{dw}{dl} = (\varepsilon - 3\psi - 4w)w, \qquad (15)$$

$$\frac{dv}{dl} = (2\rho - \psi)v, \qquad (16)$$

$$\frac{dr}{dl} = (2-\psi)r + \frac{w}{(1+r)^2}, \qquad (17)$$

$$\frac{dh}{dl} = \frac{d+2-\psi}{2} h + \frac{w}{g(1+r)}. \qquad (18)$$

It follows from the nonanalyticity of the long-range term of the Hamiltonian (1) that v does not acquire any contribution by the elimination step and changes only under the rescaling. Thus, the second equation (16) is exact, whereas the other three are correct only to one-loop order.

The last two equations (17) and (18) show in general fixed points r^*, $h^* = O(w^*)$ for the relevant parameters r and h. Using these values the first two equations (15),(16) in combination with ψ (14) lead to four different fixed points for the coupling constants w and v. There are two Gaussian ones ($w^*=0$), namely, a short-range fixed point with $v^*=0$, stable for $\varepsilon<0$, $\rho<0$ with $\eta:=\psi^*=0$, and a long-range fixed point with $v^* = \eta = 2\rho$, stable for $\varepsilon<6\rho$, $\rho>0$. Beside these trivial fixed points there are two nontrivial ones with $w^*>0$. The well-known [13,14] short-range fixed point with $v^*=0$ follows from Eq. (15) as $w^* = \varepsilon/3 + O(\varepsilon^2)$ and leads to $\eta = \eta_{SR} = -\varepsilon/9 + O(\varepsilon^2)$. It is stable for $\varepsilon>0$ as long as $2\rho < \eta_{SR}$. But if $2\rho > \eta_{SR}$, it becomes unstable and the long-range fixed point develops from Eq. (16) with $v^* = 2\rho + (\varepsilon-6\rho)/12 + O(\varepsilon^2)$, $w^* = (\varepsilon-6\rho)/4 + O(\varepsilon^2)$, and is stable up to $\rho = \varepsilon/6$, and $\eta = \eta_{LR} = 2\rho$. The stability regions are shown in Fig. 1. In each case the short-range behavior changes *continuously* to the long-range behavior and vice versa at the line defined by $\eta_{SR} = \eta_{LR}$. For all fixed points, the correlation length exponent ν follows from the
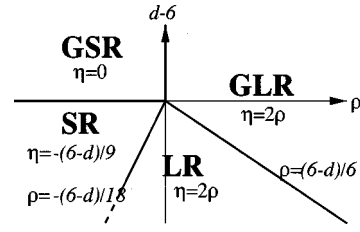

FIG. 1. Stability regions of long- and short-range behavior. Nontrivial short- (SR) and long-range (LR), as well as trivial Gaussian short- (GSR) and long-range (GLR) regions are shown. The behavior changes *continuously* at the boundaries.

linearized equation (17) for $r - r^*$. Here, in the case of the Yang-Lee model, a Ward identity states $\beta=1$ for the order parameter exponent [9,14] and one obtains $\nu = 2/(d-2+\eta)$ for both nontrivial fixed points.

To get a picture of the renormalization flow of the coupling constants and the movement of the fixed points for $\varepsilon > 0$, I have rescaled the variables as $x = w/\varepsilon$, $y = v/\varepsilon$, and introduced the parameter $p = 2\rho/\varepsilon$. With a "time" $t = l/\varepsilon$ the equations of motion are found as

$$\dot{x} = (1 - 3x - 3y)x,$$

$$\dot{y} = \left(p + \frac{x}{3} - y \right) y. \qquad (19)$$

The flows and the fixed points are shown in Fig. 2 for different parameter values. Again one sees the *continuous* bifurcation of the different fixed points corresponding to the *continuous* crossover of short-range and long-range behavior.

In conclusion I have shown that an old result concerning the crossover between long- and short-range interaction behavior in critical systems with negative short-range Fisher exponent η_{SR} is incorrect. As in the case of a positive Fisher exponent, the behavior changes continuously at a line de-
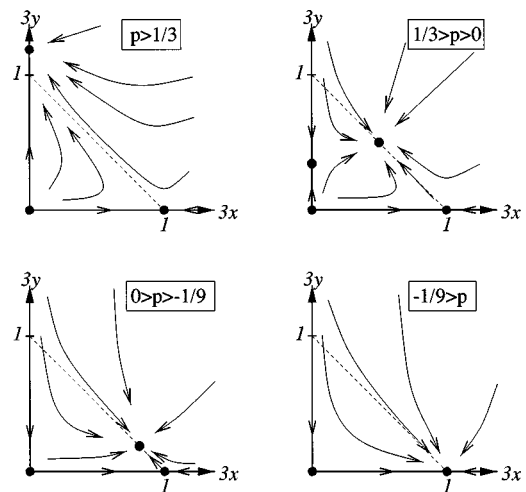

FIG. 2. Renormalization flow of the coupling constants $x = w/\varepsilon$, $y = v/\varepsilon$ for different parameter values $p = 2\kappa/\varepsilon$ in the case of $\varepsilon = 6 - d > 0$ to one-loop order. The topology of the flow changes *continuously* with p, and there exists always a stable long-range ($y>0$) fixed point if $p > -1/9$.

fined by $\eta_{SR} = \eta_{LR} = 2 - \sigma$. This general result is in accordance with previous findings in the special case $d=1$ obtained by real space renormalization [19]. The long-range interactions dominate always as long as $\eta_{SR} < \eta_{LR}$.

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